An intermediate experiment with a lossy transmission line

Thomas B. Greenslade, Jr.
Department of Physics, Kenyon College, Gambier, Ohio 43022

(Received 27 August 1987; accepted for publication 8 April 1988)

The study of a model of the lossy transmission line, whose repeating elements are a series resistance and a shunt capacitance, leads students to a surprising number of ideas in physics. The author has used the system to introduce students to analog simulation, and to the ideas of Fourier synthesis, exponential decay, dispersion, and phase shifts. The analysis of the system is well known, although the specific method of analysis discussed in this note seems not to have been published before.

I. THE MODEL

The outline presented here of the analysis of the general transmission line is based on Bronwell's treatment.[1] In Fig. 1(a), the transmission line is represented by a series resistance per unit length R; a series inductance per unit length L; a leakage resistance per unit length $1/Y$; and a shunt capacitance per unit length C. The potential difference ΔV between the ends of a short segment of length Δx, which carries current I, is due to the IR drop along the segment and the back emf,

$$\Delta V = \frac{\partial V}{\partial x}\Delta x = IR\,\Delta x + L\,\Delta x\frac{\partial I}{\partial t}.$$

The change in the series current ΔI between the two ends of the segment is due to leakage and the charging of the shunt capacitance,

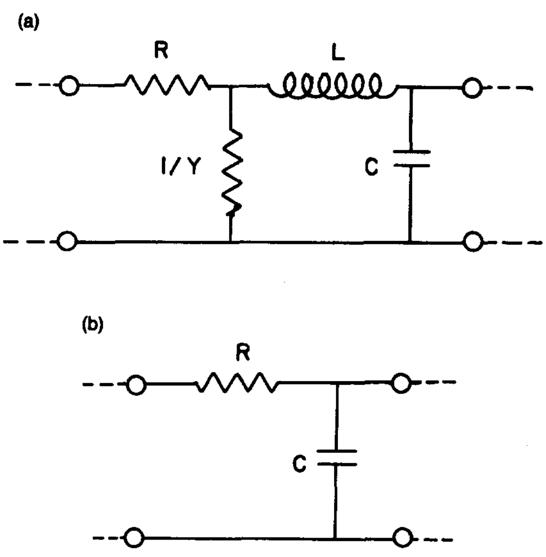

Fig. 1. (a) One segment of the generalized model for the transmission line. (b) The model for a transmission line having only series resistance and shunt capacitance.

$$\Delta I = \frac{\partial I}{\partial x}\Delta x = Y\Delta x\,V + C\,\Delta x\frac{\partial V}{\partial t}.$$

Canceling out the arbitrary segment length and using cross differentiation to obtain a general wave equation for V gives

$$\frac{\partial^2 V}{\partial x^2} = RYV + (RC + LY)\frac{\partial V}{\partial t} + LC\frac{\partial^2 V}{\partial t^2}.$$

We are interested in the case shown in Fig. 1(b), in which the inductance and leakage effects are small. If the terms with the coefficients Y and L are omitted, the working equation for the experiment is obtained,

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}. \quad (1)$$

This is a wave equation, but the solutions will be seen to display exponential decay as the wave travels down the line, and also dispersion and phase shifts.

II. THE THERMAL ANALOG

The same form is also displayed by the diffusion equation, which is followed by several systems, particularly the flow of heat down a semi-infinite bar wrapped in thermal insulation. In this case, the dependent variable is the temperature, which is a function of the time and the position along the bar as follows:

$$\frac{\partial^2 T}{\partial x^2} = \frac{CD}{K}\frac{\partial T}{\partial t}. \quad (2)$$

Here, K is the coefficient of thermal conductivity, C is the specific heat capacity, and D is the density of the material; the combination K/CD is often called the diffusivity.[2]

The isomorphism of Eqs. (1) and (2) is often used to permit the thermal situation, which is hard to set up, to be simulated by the electrical analog. A generalized treatment has been given by Karplus.[3] Steere[4] has shown how to use the analog to study heat flow in two-dimensional systems, and Tomlin and Fullarton[5] have applied the transmission line analogy to the study of one-dimensional heat flow problems with long time constants.

III. THE EXPERIMENT

In the present experiment, the emphasis is placed on the observation of the solutions to Eq. (2). The signal applied to one end of the line is assumed to be sinusoidal with angular frequency ω. It is reasonable to assume that the time variation at points down the line will also have the same form. The space dependence of the signal is assumed to be a decaying exponential, since there is a lossy series element R present. Hence, the trial solution is

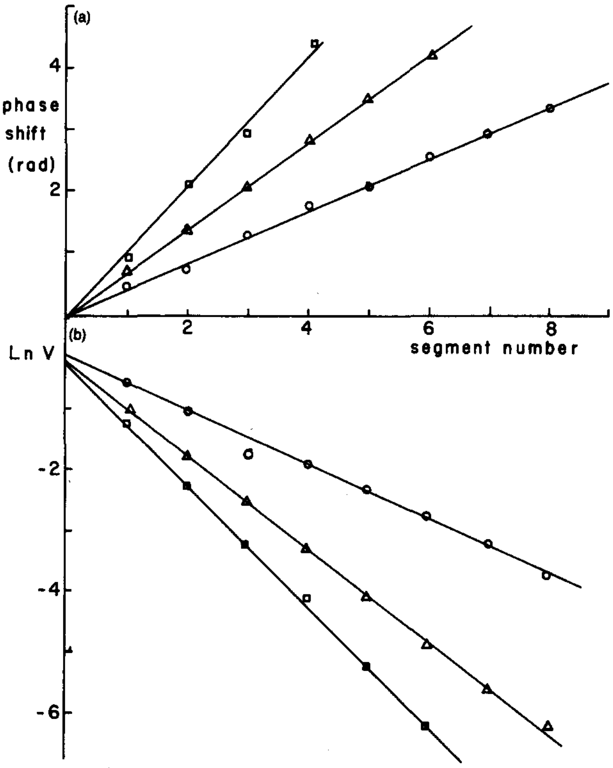

Fig. 2. (a) Phase shift as a function of segment number for the lossy transmission line. (b) Semilog plot of relative amplitude as a function of segment number. Circles are for a 100-Hz sine wave, triangles for a 300-Hz sine wave, and squares for a 500-Hz sine wave.

$$V = V_0 \exp[\,j(\omega t - bx)\,] .$$

Putting this into Eq. (1) shows that the coefficient b must be complex, and the proper solution is

$$V = V_0 \exp(-\sqrt{\omega RC/2}x)\exp[\,j(\omega t - \sqrt{\omega RC/2}x)\,] . \tag{3}$$

Here, x is the distance in units of segments down the transmission line, measured from the driven end.

The first term in the solution indicates that the amplitude of the signal is attenuated as it passes down the line, while the second term describes the oscillatory nature of the solution. The ratio of the amplitudes of the nth to the $(n-1)$th segments is $e^{-\theta}$, where $\theta = (\omega RC/2)^{1/2}$, and it can be seen that, at a given time, the signals at adjacent segments differ in phase by an angle of θ rad.

The system also displays the phenomenon of dispersion. Equation (3) shows that the wavenumber is $(\omega RC/2)^{1/2}$. Thus the phase velocity is $v = \omega/k = (2\omega/RC)^{1/2}$; in this case, the higher frequency components of a complex periodic wave would travel faster down the transmission line.

In a typical experiment, a 15-segment transmission line was set up on a bread board, using 2.7-kΩ resistors and 0.22-μF capacitors. Sinusoidal signals of frequencies 100, 300, and 500 Hz were applied to the end of the line, and the amplitude and phase shifts measured as far as possible down the line. A dual-beam oscilloscope was used to observe the amplitude and phase shift of the signals at each segment, relative to the input signal. The phase shifts were measured (a) by observing the horizontal displacement of the attenuated signal relative to the input signal when the two traces were superimposed on the CRO screen, and (b) by observing the shape of the Lissajous figures formed by the input and attenuated signals.

Table I. Values of $\theta = (\omega RC/2)^{1/2}$.

	Theory	Attenuation	Phase shift
f = 100 Hz	0.43 ± 0.02	0.433 ± 0.003	0.422 ± 0.008
f = 300 Hz	0.75 ± 0.04	0.752 ± 0.007	0.703 ± 0.004
f = 500 Hz	0.97 ± 0.05	0.99 ± 0.01	1.08 ± 0.06

Figure 2(a) shows typical data for phase shift as a function of segment number for the three frequencies and Fig. 2(b) is a semilog plot of the signal amplitude as a function of segment number. For a given frequency, the absolute values of the slopes of the two lines should be equal to the quantity θ. Table I shows that within the limits of experimental error this is true. The uncertainty in the slopes was obtained from a least-squares curve-fitting program, and the limiting factor in the theoretical value of θ was assumed to be the 10% uncertainty in the value of the capacitance.

This experiment is used in a course on oscillations and waves for second semester sophomores. Fourier techniques have already been discussed, so that the students know the Fourier coefficients for a square wave (odd harmonics only, with the coefficients inversely proportional to the harmonic number). They are asked to apply a 100-Hz square wave to the input of the transmission line, and observe the wave shape at various points down the line. The shape of the wave degenerates for two reasons: The amplitudes of the higher frequency components decay away more rapidly, and these components undergo increasingly larger phase shifts. Eventually, a point is reached where the wave shape cannot be distinguished from a sinusoid. The students are then asked to go back to the data in Fig. 2(b) to see if the contributions of the third and fifth harmonics have become small enough relative to the fundamental to produce the sinusoidal wave shape.

[1]Arthur Bronwell, *Advanced Mathematics in Physics and Engineering* (McGraw-Hill, New York, 1953), pp. 161–163.
[2]Reference 1, pp. 254–256.
[3]Walter J. Karplus, *Analog Simulation* (McGraw-Hill, New York, 1958), pp. 204–209.
[4]R. C. Steere, "Solution of two-dimensional transient heat flow problems by electrical analogue," Phys. Educ. **6**, 443–447 (1971).
[5]D. H. Tomlin and G. K. Fullerton, "Electrical circuit analogues of thermal conduction and diffusion," Phys. Educ. **13**, 295–299 (1978).